

Non-Abelian Discrete Family Symmetries of Leptons and Quarks

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Abstract

Recent progress is reviewed regarding the application of the non-Abelian discrete symmetries S_3 , D_4 , and A_4 to the understanding of family structure in leptons and quarks.

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1 Introduction

The Standard Model (SM) has 3 families of quarks and leptons. In the notation where all fermions are left-handed, they are $Q_i = (u_i, d_i)$, u_i^c, d_i^c ; $L_i = (\nu_i, l_i)$, l_i^c , $[i = 1, 2, 3]$. There is also one scalar Higgs doublet $\Phi = (\phi^+, \phi^0)$. Under the $SU(2)_L \times U(1)_Y$ gauge symmetry, the allowed Yukawa couplings are $h_{ij}^u(d_i\phi^+ + u_i\phi^0)u_j^c$, $h_{ij}^d(d_i\bar{\phi}^0 - u_i\phi^-)d_j^c$, and $f_{ij}(l_i\bar{\phi}^0 - \nu_i\phi^-)l_j^c$. Let $\langle\phi^0\rangle = v$, then

$$\mathcal{M}_u = h_{ij}^u v = V_L^u \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} (V_R^u)^\dagger, \quad (1)$$

$$\mathcal{M}_d = h_{ij}^d v^* = V_L^d \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} (V_R^d)^\dagger. \quad (2)$$

The observed quark mixing matrix is

$$V_{CKM} = (V_L^u)^\dagger V_L^d \simeq \begin{pmatrix} 0.976 & 0.22 & 0.003 \\ -0.22 & 0.98 & 0.04 \\ 0.007 & -0.04 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

Similarly,

$$\mathcal{M}_l = f_{ij} v^* = U_L^l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_R^l)^\dagger, \quad (4)$$

whereas the neutrino mass matrix is either (A) Dirac, in which case,

$$(A) : \mathcal{M}_\nu^D = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_R^\nu)^\dagger, \quad (5)$$

or (B) Majorana, in which case,

$$(B) : \mathcal{M}_\nu^M = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_L^\nu)^T. \quad (6)$$

[Note: Neutrinos still do not have their own names!] The observed lepton mixing matrix is

$$U_{MNS} = (U_L^l)^\dagger U_L^\nu \simeq \begin{pmatrix} 0.85 & 0.52 & 0.053 \\ -0.33 & 0.62 & -0.72 \\ -0.40 & 0.59 & 0.70 \end{pmatrix} \simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \quad (7)$$

It is clear that U_{MNS} is very different from V_{CKM} and the study of non-Abelian discrete family symmetries may help us understand why.

Since the minimal SM does not contain ν^c , we should first understand how neutrinos get mass. I will assume at the outset that at low energy, ν^c is indeed absent, in which case the only way that neutrinos can obtain mass is through the unique effective dimension-five operator first written down by Weinberg [1] 25 years ago:

$$\frac{f_{ij}}{2\Lambda} \nu_i \nu_j \phi^0 \phi^0 \Rightarrow (\mathcal{M}_\nu)_{ij} = f_{ij} \frac{\langle \phi^0 \rangle^2}{\Lambda}. \quad (8)$$

There are 2 things to notice in the above. (1) These masses are Majorana. (2) These masses are necessarily “seesaw” in form, because Λ is a large effective mass.

To understand how this operator may be realized [2], consider the 6 possible pairwise products of L_i and Φ in the SM, as shown below.

Table 1: Pairwise Products of L_i and Φ .

	$SU(2)_L$ singlets		$SU(2)_L$ triplets
(A)	$\nu_i \phi^0 - l_i \phi^+$	(D)	$[\nu_i \phi^+, (\nu_i \phi^0 + l_i \phi^+)/\sqrt{2}, l_i \phi^0]$
(B)	$\nu_i l_j - l_i \nu_j$	(E)	$[\nu_i \nu_j, (\nu_i l_j + l_i \nu_j)/\sqrt{2}, l_i l_j]$
(C)	$\phi_1^+ \phi_2^0 - \phi_1^0 \phi_2^+$	(F)	$[\phi^+ \phi^+, \sqrt{2} \phi^+ \phi^0, \phi^0 \phi^0]$

There are 4 basic combinations to obtain the effective operator of Eq. (8).

(I) Consider (A) \times (A). The connection must be a neutral fermion singlet N with a large Majorana mass, i.e. the canonical seesaw mechanism with $m_\nu \sim \langle \phi^0 \rangle^2 / m_N$.

(II) Consider (E) \times (F). The connection is now a scalar triplet (ξ^{++}, ξ^+, ξ^0) with $m_\nu \sim \mu \langle \phi^0 \rangle^2 / m_\xi^2$, where μ is the trilinear coupling of ξ to (F).

(III) Consider (D) \times (D). The connection is a Majorana fermion triplet $(\Sigma^+, \Sigma^0, \Sigma^-)$ with $m_\nu \sim \langle \phi^0 \rangle^2 / m_\Sigma$.

These 3 cases are all the possible [2] tree-level realizations of Eq. (8). (I) has dominated the literature, but (II) is getting increasingly more attention.

(IV) Consider (B) \times (C). This requires 2 Higgs doublets and a charged scalar singlet η^+ for the connection, i.e. the well-known one-loop radiative mechanism first proposed by Zee [3].

Another important question is the scale of m_N . If $m_\nu \sim 1$ eV is assumed, then $m_N \sim 10^{13}$ GeV $(\langle \phi^0 \rangle / 10^2 \text{ GeV})^2$. However, suppose there exists a symmetry which forbids N from coupling to ϕ^0 but allows it to couple to η^0 instead, with $\langle \eta^0 \rangle \sim 1$ MeV, then $m_N \sim 1$ TeV may be obtained.

Here is the simplest example [4]. Assign lepton number $L = 0$ to N and (ϕ^+, ϕ^0) , but $L = -1$ to (η^+, η^0) , then $\nu N \phi^0$ is forbidden, but $\nu N \eta^0$ is allowed. Now break L softly by the term $\mu_{12}^2 (\Phi^\dagger \eta + \eta^\dagger \Phi)$ in the Higgs potential, then

$$\langle \eta^0 \rangle \sim \mu_{12}^2 \langle \phi^0 \rangle / m_\eta^2. \quad (9)$$

Let $m_\eta \sim 1$ TeV, $\langle \phi^0 \rangle \sim 10^2$ GeV, $\mu_{12}^2 \sim 10 \text{ GeV}^2$, then $\langle \eta^0 \rangle \sim 1$ MeV as desired. This scenario is actually realized in the well-known case of R -parity violating supersymmetry, where $\langle \bar{\eta}^0 \rangle$ is $\langle \tilde{\nu} \rangle$ and m_N is an effective neutralino mass. Having m_N at the TeV scale makes the seesaw mechanism testable at the LHC.

2 Some non-Abelian discrete groups

The simplest non-Abelian discrete symmetry is S_3 , the permutation of 3 objects, which is also the symmetry group of the equilateral triangle. It has 6 elements in 3 equivalence classes: $[C_1] : (123)$, $[C_2] : (231), (312)$, and $[C_3] : (132), (321), (213)$. Its character table is given below.

Table 2: Character Table of S_3 .

class	n	h	χ_1	χ_2	χ_3
C_1	1	1	1	1	2
C_2	2	3	1	1	-1
C_3	3	2	1	-1	0

Here n is the number of elements and h is the order of each element. The character of each representation is its trace and must satisfy the following two orthogonality conditions:

$$\sum_{C_i} n_i \chi_{ai} \chi_{bi}^* = n \delta_{ab}, \quad \sum_{\chi_a} n_i \chi_{ai} \chi_{aj}^* = n \delta_{ij}, \quad (10)$$

where n is the total number of elements. The number of irreducible representations must be equal to the number of equivalence classes.

Table 3: Character Table of D_4 .

class	n	h	χ_1	χ_2	χ_3	χ_4	χ_5
C_1	1	1	1	1	1	1	2
C_2	1	2	1	1	1	1	-2
C_3	2	4	1	-1	-1	1	0
C_4	2	2	1	1	-1	-1	0
C_5	2	2	1	-1	1	-1	0

A slightly bigger group is D_4 , the symmetry group of the square. It has 8 elements in

5 equivalence classes: $[C_1] : (1234)$, $[C_2] : (3412)$, $[C_3] : (2341), (4123)$, $[C_4] : (1432), (3214)$, $[C_5] : (2143), (4321)$. Its character table is given above.

Another interesting group is A_4 , the even permutation of 4 objects, which is also the symmetry group of the tetrahedron. It has 12 elements in 4 equivalence classes: $[C_1] : (1234)$, $[C_2] : (1342), (4213), (2431), (3124)$, $[C_3] : (1423), (3241), (4132), (2314)$, $[C_4] : (2143), (3412), (4321)$. Defining

$$\omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad (11)$$

its character table is given below.

Table 4: Character Table of A_4 .

class	n	h	χ_1	χ_2	χ_3	χ_4
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	-1

Around the year 390 BCE, the Greek mathematician Theaetetus proved that there were 5 and only 5 perfect geometric solids. This posed a puzzle to Plato because only 4 basic elements (fire, air, water, and earth) were known, so a fifth element must be missing and Plato called it “quintessence” which is supposed to pervade the cosmos. He also assigned each perfect geometric solid to these elements, as shown below.

This was of course the first theory of everything (TOE)! In the early 19th century, group theory was developed and each solid was identified to have the symmetries indicated: S_4 is the group of permutation of 4 objects and A_5 is the even permutation of 5 objects. Two pairs of solids are dual to one another because each can be perfectly embedded into the other and vice versa. The tetrahedron is special because it is self-dual. Compare this to today’s TOE, i.e. string theory. There are 5 consistent string theories in 10 dimensions, which are

Table 5: Five Perfect Geometric Solids.

solid	faces	vertices	Plato	group
tetrahedron	4	4	fire	A_4
octahedron	8	6	air	S_4
icosahedron	20	12	water	A_5
hexahedron	6	8	earth	S_4
dodecahedron	12	20	quintessence	A_5

related by S,T,U dualities: Type I is dual to Heterotic $SO(32)$, Type IIA is dual to Heterotic $E_8 \times E_8$, and Type IIB is self-dual. Just as the 5 perfect geometric solids may be embedded in a sphere, the 5 string theories may be embedded in a single underlying M theory.

3 Representations of S_3 , D_4 , A_4

From Table 2, we see that S_3 has 3 irreducible representations: **1**, **1'**, and **2**. This group was used already in 1964 by Yamaguchi [5] for strong interactions, just before $SU(3)$ (the eightfold way) was proposed by Gell-Mann. Subsequently, it was applied to the 2×2 quark mass matrix by Pakvasa and Sugawara [6]. It has been studied by many authors, using a 2×2 real representation as follows. Put the vertices of the equilateral triangle in the (x, y) plane at $1 \sim (1, 0)$, $2 \sim (-1/2, \sqrt{3}/2)$, $3 \sim (-1/2, -\sqrt{3}/2)$. The 6 group elements are then represented by the corresponding matrices which permute these vertices. However, there is a better way. Let the 3 vertices be denoted instead by $(x + iy, x - iy)$, then $1 \sim (1, 1)$, $2 \sim (\omega, \omega^2)$, $3 \sim (\omega^2, \omega)$, where ω is the same quantity defined in Eq. (11). This “complex” representation [7] is of course related to the real representation by a unitary transformation, i.e.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} (Real) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = (Complex), \quad (12)$$

but is much more convenient in model building, as shown below.

Table 6: Real and “Complex” Representations of S_3 .

class	element	real rep	complex rep
C_1	(123)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_2	(231)	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$
C_2	(312)	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$
C_3	(132)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
C_3	(321)	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$
C_3	(213)	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$

The basic S_3 group multiplication rule is $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}$, but different combinations of the doublet components appear in the decomposition according to which representation is used, as shown below.

Table 7: $\mathbf{2} \times \mathbf{2}$ Decompositions of S_3 .

rep	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$
real	11+22	12 − 21	$\begin{pmatrix} 12 + 21 \\ 11 - 22 \end{pmatrix}$
complex	12+21	12 − 21	$\begin{pmatrix} 22 \\ 11 \end{pmatrix}$

If $(\psi_1, \psi_2) \sim \mathbf{2}$, then in the real representation, $(\psi_1^*, \psi_2^*) \sim \mathbf{2}$, but in the complex representation, $(\psi_2^*, \psi_1^*) \sim \mathbf{2}$ instead. The invariant product of 3 doublets in the former is given by $121 + 211 + 112 - 222$, whereas in the latter, it is simply $111 + 222$.

From Table 3, we see that D_4 has 5 irreducible representations: $\mathbf{1}^{++}$, $\mathbf{1}^{+-}$, $\mathbf{1}^{-+}$, $\mathbf{1}^{--}$, and $\mathbf{2}$. The 8 elements may be represented by the eight 2×2 matrices ± 1 , $\pm i\sigma_2$, $\pm\sigma_1$, $\pm\sigma_3$.

The basic group multiplication rule is

$$\mathbf{2} \times \mathbf{2} = \mathbf{1}^{++}(11 + 22) + \mathbf{1}^{+-}(12 + 21) + \mathbf{1}^{-+}(11 - 22) + \mathbf{1}^{--}(12 - 21). \quad (13)$$

Suppose we replace $\pm\sigma_{1,3}$ by $\pm i\sigma_{1,3}$, then we obtain a different group of 8 elements with the same character table, except $h = 4$ (not 2) for $C_{4,5}$ in Table 3. This is the quaternion group Q_8 such that $(\psi_2^*, -\psi_1^*) \sim \mathbf{2}$ if $(\psi_1, \psi_2) \sim \mathbf{2}$, and

$$\mathbf{2} \times \mathbf{2} = \mathbf{1}^{++}(12 - 21) + \mathbf{1}^{+-}(11 - 22) + \mathbf{1}^{-+}(12 + 21) + \mathbf{1}^{--}(11 + 22). \quad (14)$$

Lastly from Table 4, we see that A_4 has 4 irreducible representations: $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, and $\mathbf{3}$, with the following multiplication rule:

$$\begin{aligned} \mathbf{3} \times \mathbf{3} = & \mathbf{1}(11 + 22 + 33) + \mathbf{1}'(11 + \omega^2 22 + \omega 33) + \mathbf{1}''(11 + \omega 22 + \omega^2 33) \\ & + \mathbf{3}(23, 31, 12) + \mathbf{3}(32, 13, 21). \end{aligned} \quad (15)$$

4 S_3 models

There are two recently proposed S_3 models of leptons and quarks. Both are successful phenomenologically, but they differ in their choice of representations and make different predictions regarding the neutrino mass matrix. The first one [8] used the conventional real representation and assigned 3 families of quarks, leptons (including singlet neutrinos ν_i^c), and Higgs doublets all to $\mathbf{1} + \mathbf{2}$ of S_3 . Assuming first equal VEVs of the 2 Higgs doublets in $\mathbf{2}$, there are then 5 parameters in each Dirac mass matrix. Adding an extra Z_2 symmetry such that ν_3^c and Φ_3 are odd, while all other fields are even, 2 parameters each in \mathcal{M}_D^l and \mathcal{M}_D^q are eliminated. This leads to maximal $\nu_\mu - \nu_\tau$ mixing in an inverted hierarchy of neutrino masses, predicting $U_{e3} \simeq -3.4 \times 10^{-3}$.

A more recent model [9] used the complex representation and assigned leptons and quarks differently from their charge conjugates. Singlet neutrinos are not used. Consider first the

2nd and 3rd families. Assign $[(\nu_2, l_2), (\nu_3, l_3)] \sim \mathbf{2}$, $l_2^c \sim \mathbf{1}$, $l_3^c \sim \mathbf{1}'$, and similarly for quarks.

Add $[\Phi_1, \Phi_2] \sim \mathbf{2}$, then

$$\mathcal{M}_l = \begin{pmatrix} f_1 v_2 & -f_2 v_2 \\ f_1 v_1 & f_2 v_1 \end{pmatrix}, \quad \mathcal{M}_d = \begin{pmatrix} h_1^d v_2 & -h_2^d v_2 \\ h_1^d v_1 & h_2^d v_1 \end{pmatrix}, \quad \mathcal{M}_u = \begin{pmatrix} h_1^u v_1^* & -h_2^u v_1^* \\ h_1^u v_2^* & h_2^u v_2^* \end{pmatrix}. \quad (16)$$

Let $v_{1,2} \simeq v$, then $m_s \simeq \sqrt{2}h_1^d v$, $m_b \simeq \sqrt{2}h_2^d v$, $m_c \simeq \sqrt{2}h_1^u v^*$, $m_t \simeq \sqrt{2}h_2^u v^*$, and

$$\theta_{23}^q \simeq \frac{|v_2|^2 - |v_1|^2}{|v_2|^2 + |v_1|^2} \simeq 2 \frac{|v_2| - |v_1|}{|v_2| + |v_1|} \simeq 0.04. \quad (17)$$

Neutrino masses are assumed to be Majorana and come from heavy Higgs triplets $[\xi_1, \xi_2] \sim \mathbf{2}$.

Thus

$$\mathcal{M}_\nu = \begin{pmatrix} f_3 u_1 & 0 \\ 0 & f_3 u_2 \end{pmatrix}. \quad (18)$$

For $u_1 \neq u_2$, this implies a mismatch with \mathcal{M}_l of

$$\theta_{23}^l \simeq \frac{\pi}{4} - \frac{1}{2}\theta_{23}^q \simeq \frac{\pi}{4}. \quad (19)$$

The first family is then added as a ‘‘perturbation’’: $(\nu_1, l_1), l_1^c, \Phi_3 \sim \mathbf{1}$, and similarly for quarks. As a result,

$$\theta_{13}^l \simeq \frac{1}{2} \sin 2\theta_{12}^l \frac{\Delta m_{sol}^2}{\Delta m_{atm}^2}, \quad (20)$$

i.e. $0.008 < \theta_{13}^l < 0.032$ is predicted in a normal hierarchy of neutrino masses.

5 D_4 models

The symmetry D_4 has recently been applied [10] to leptons (including ν_i^c). They are assigned as $\mathbf{1}^{++} + \mathbf{2}$ under D_4 . The Higgs sector has 3 doublets: $\Phi_{1,2,3} \sim \mathbf{1}^{++}, \mathbf{1}^{++}, \mathbf{1}^{-+}$, and 2 singlets: $\chi_{1,2} \sim \mathbf{2}$. Adding an extra Z_2 symmetry such that $l_1^c, \nu_{1,2,3}^c, \Phi_1$ are odd, while all other fields are even, the charged-lepton mass matrix is diagonal with 3 independent eigenvalues and the neutrino mass matrices are given by

$$\mathcal{M}_\nu^D = \begin{pmatrix} y_1 v_1^* & 0 & 0 \\ 0 & y_2 v_1^* & 0 \\ 0 & 0 & y_2 v_1^* \end{pmatrix}, \quad \mathcal{M}_{\nu^c}^M = \begin{pmatrix} M & y_\chi u_1 & y_\chi u_2 \\ y_\chi u_1 & M' & 0 \\ y_\chi u_2 & 0 & M' \end{pmatrix}. \quad (21)$$

Assuming $u_1 = u_2$, we then have

$$\mathcal{M}_\nu^M = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad (22)$$

which automatically yields [11]

$$U_{MNS} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta/\sqrt{2} & \cos \theta/\sqrt{2} & -1/\sqrt{2} \\ \sin \theta/\sqrt{2} & \cos \theta/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \quad (23)$$

The neutrino masses of this model are further constrained by $(\mathcal{M}_\nu^M)_{23} = 0$ which implies $|(\mathcal{M}_\nu)_{ee}| = m_1 m_2 / m_3$, with $m_1 < m_2 < m_3$, i.e. normal ordering. If $u_1 \neq u_2$, then $\theta_{23} \neq \pi/4$, but θ_{13}^l remains zero.

6 A_4 models

Just as S_3 would be ideal to describe 2 families, A_4 appears perfect for 3 families, because it has the irreducible representations $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, and $\mathbf{3}$. In particular, it is a natural choice to have 3 degenerate neutrino masses [12]. Let $(\nu_i, l_i) \sim \mathbf{3}$, $l_1^c \sim \mathbf{1}$, $l_2^c \sim \mathbf{1}'$, $l_3^c \sim \mathbf{1}''$, and $\Phi_i \sim \mathbf{3}$, $\xi \sim \mathbf{1}$, then

$$\mathcal{M}_l = \begin{pmatrix} f_1 v_1 & f_2 v_1 & f_3 v_1 \\ f_1 v_2 & f_2 \omega v_2 & f_3 \omega^2 v_2 \\ f_1 v_3 & f_2 \omega^2 v_3 & f_3 \omega v_3 \end{pmatrix}, \quad \mathcal{M}_\nu = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & m_0 \end{pmatrix}. \quad (24)$$

For $v_1 = v_2 = v_3 = v$, \mathcal{M}_l is diagonalized on the left by U_L and on the right by $U_R = 1$, where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \mathcal{M}_\nu^{(e,\mu,\tau)} = U_L^\dagger \mathcal{M}_\nu U_L^* = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & 0 & m_0 \\ 0 & m_0 & 0 \end{pmatrix}. \quad (25)$$

In the quark sector, both $\mathcal{M}_{u,d}$ are diagonalized by U_L , hence $V_{CKM} = 1$ is obtained as a first approximation.

Under the most general radiative corrections, the neutrino mass matrix takes the form [13]

$$\mathcal{M}_\nu = m_0 \begin{pmatrix} 1 + 2\delta + 2\delta' & \delta'' & \delta''^* \\ \delta'' & \delta & 1 + \delta \\ \delta''^* & 1 + \delta & \delta \end{pmatrix}, \quad (26)$$

where all complex phases have been rotated away except for that of δ'' . If δ'' happens to be real, then this mass matrix is of the form of Eq. (22), yielding U_{MNS} of Eq. (23) automatically. If δ'' has an imaginary part, then U_{e3} becomes nonzero and is approximately given by $iIm\delta''/\sqrt{2}\delta$, thus predicting maximal CP violation in neutrino oscillations. The appropriate nonzero radiative corrections may be obtained in the context of supersymmetry [14], which is known to be also viable for generating a realistic V_{CKM} [15].

The above A_4 model predicts nearly degenerate neutrino masses, with Δm_{atm}^2 coming from radiative corrections. This means that the common mass m_0 of neutrinos should not be much greater than about 0.3 eV, which is also the upper limit from neutrinoless double beta decay and cosmological observations. If $m_0 < 0.1$ eV is established in the future, this model can be ruled out. On the other hand, the A_4 model may be modified to accept arbitrary neutrino masses [16]. Instead of using only $\xi_1 \sim \mathbf{1}$, consider the addition of $\xi_2 \sim \mathbf{1}'$, $\xi_3 \sim \mathbf{1}''$, and $\xi_{4,5,6} \sim \mathbf{3}$, with $\langle \xi_{1,2,3,4}^0 \rangle \neq 0$, then

$$\begin{aligned} \mathcal{M}_\nu^{(e,\mu,\tau)} &= U_L^\dagger \begin{pmatrix} a+b+c & 0 & 0 \\ 0 & a+\omega b+\omega^2 c & d \\ 0 & d & a+\omega^2 b+\omega c \end{pmatrix} U_L^* \\ &= \begin{pmatrix} a+(2d/3) & d-(d/3) & c-(d/3) \\ b-(d/3) & c+(2d/3) & a-(d/3) \\ c-(d/3) & a-(d/3) & b+(2d/3) \end{pmatrix}, \end{aligned} \quad (27)$$

where a, b, c, d come from $\langle \xi_{1,2,3,4}^0 \rangle$ respectively. If $b = c$, then the eigenvalues of this mass matrix are $m_1 = a - b + d$, $m_2 = a + 2b$, $m_3 = -a + b + d$, and

$$U_{MNS} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \quad (28)$$

which is exactly the conjecture of Eq. (7). This predicts $\sin^2 2\theta_{atm} = 1$, $U_{e3} = 0$, and $\tan^2 \theta_{sol} = 0.5$, for arbitrary $m_{1,2,3}$. Experimentally, solar neutrino oscillation data prefer $\tan^2 \theta_{sol}$ in the vicinity of 0.4, whereas this model allows only the range 0.50 to 0.52 even if $b \neq c$. On the other hand, if there is new physics at the TeV scale such as supersymmetry, this value may be shifted.

7 Generic phenomenological consequences

Each model discussed so far has 3 Higgs doublets. In general, this implies tree-level flavor-changing charged-lepton interactions, unless \mathcal{M}_l is already diagonal, as in D_4 . In any case, the off-diagonal Yukawa couplings are also suppressed by charged-lepton masses, and are usually safe phenomenologically. Generically, many scalar particles should become observable at the LHC, decaying into various leptons.

Consider the specific case of the A_4 model with 3 Higgs doublets. The Higgs potential is given by [12]

$$\begin{aligned} V = & m^2 \sum_i \Phi_i^\dagger \Phi_i + \frac{1}{2} \lambda_1 (\sum_i \Phi_i^\dagger \Phi_i)^2 + \frac{1}{2} \lambda_2 \sum_{i,j} (3\delta_{i,j} - 1) (\Phi_i^\dagger \Phi_i) (\Phi_j^\dagger \Phi_j) \\ & + \frac{1}{2} \lambda_3 \sum_{i \neq j} (\Phi_i^\dagger \Phi_j) (\Phi_j^\dagger \Phi_i) + \frac{1}{2} \lambda_4 \sum_{i \neq j} (\Phi_i^\dagger \Phi_j)^2. \end{aligned} \quad (29)$$

It has a minimum at $v = v_1 = v_2 = v_3 = (-m^2/(3\lambda_1 + 2\lambda_3 + 2\lambda_4))^{1/2}$. The 3×3 mass matrices of $Re\phi_{1,2,3}^0$, $Im\phi_{1,2,3}^0$, $\phi_{1,2,3}^\pm$ are all of the form

$$\mathcal{M}^2 = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}, \quad (30)$$

with different values of a, b in different sectors. Hence the linear combination $(\phi_1 + \phi_2 + \phi_3)/\sqrt{3}$ behaves like the one Higgs doublet of the SM, and has eigenvalue $m^2 = a + 2b$, whereas the two orthogonal combinations Φ', Φ'' are degenerate in mass and have eigenvalue $m^2 = a - b$.

This means that the Higgs sector has a residual S_3 symmetry, but S_3 is not a subgroup of A_4 , so where does it come from? The answer is that V is actually also invariant under S_4 .

The Yukawa couplings of Φ' and Φ'' in this model are completely determined by the charged-lepton masses. They are given by

$$\begin{aligned}\mathcal{L}_{int} = & \frac{m_\tau}{v}[(\overline{\nu_e}, e)_L \Phi' + (\overline{\nu_\mu}, \mu)_L \Phi'']\tau_R + \frac{m_\mu}{v}[(\overline{\nu_\tau}, \tau)_L \Phi' + (\overline{\nu_e}, e)_L \Phi'']\mu_R \\ & + \frac{m_e}{v}[(\overline{\nu_\mu}, \mu)_L \Phi' + (\overline{\nu_\tau}, \tau)_L \Phi'']e_R + H.c.,\end{aligned}\tag{31}$$

which breaks S_3 explicitly.

8 Concluding remarks

If there is a family symmetry behind lepton (and quark) mass matrices, it is only evident if we know the complete Lagrangian, including the extra scalar (and possibly other) fields required to support this symmetry. With our present incomplete knowledge, we can only assume that there is such a symmetry and try to find it with some educated guesses. In the context of the Standard Model, the Higgs sector must be enlarged.

Texture zeros usually lead to relationships among mixing angles and mass ratios and they can be realized by arbitrary Abelian discrete symmetries [17] supported by a possibly large number of scalars. Non-Abelian discrete symmetries are very restrictive, with just a few representations. They may also lead to texture zeros as well as the exact equality of mass matrix elements which is impossible with an Abelian symmetry. Mixing angles may also turn out to be unrelated to masses. In the D_4 model for example, $\theta_{13} = 0$, $\theta_{23} = \pi/4$, and θ_{12} is arbitrary, independent of $m_{1,2,3}$ and $m_{e,\mu,\tau}$.

Much work has been spent on \mathcal{M}_l , \mathcal{M}_ν , and U_{MNS} in the past few years. Armed with this experience, we should go back and tackle the old problem of \mathcal{M}_u , \mathcal{M}_d , and V_{CKM} with hopefully a fresh perspective.

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